Math 524 Exam 1 Solutions

1. State the eight axioms of a vector space.

   Vector addition is an abelian group. That is, addition is commutative, associative, with an identity and inverses. More precisely:
   $x+y=y+x, x+(y+z)=(x+y)+z$ for all vectors x,y,z; there is a vector 0 such that $0+x=x$ for all x; every vector x has an associated $-x$ such that $x+(-x)=0$.
   Scalar multiplication respects the associated field. More precisely:
   For every vector x and scalars a,b, $1(x)=x, (ab)x=a(bx)$.
   Finally, two distributive laws hold. For every scalars a,b and vectors x,y,
   $a(x+y)=ax+ay, (a+b)x=ax+bx$.

2. Using only the eight vector space axioms, prove that an element of a vector space has at most one additive inverse.

   Suppose that for some vector x there were two inverses y,z. Then $x+y=0=x+z$. We have $z=0+z=(x+y)+z=(y+x)+z=y+(x+z)=y+0=0+y=y$, where we used (in order) the identity axiom, the hypothesis, commutativity, associativity, the hypothesis, commutativity, and the identity axiom again.

3. In $\mathbb{R}^2$, is it possible to have a set of two vectors that is:

   (a) independent and spanning
   (b) not independent and spanning
   (c) independent and not spanning
   (d) not independent and not spanning

   Solution 1 (theorem): Writing the two vectors as columns of a 2×2 matrix, we apply Theorem 2.5 from the text; they are independent if and only if they span $\mathbb{R}^2$. Hence (a) and (d) are possible, while (b) and (c) are not.
   Solution 2 (dimension): If two vectors were independent, the subspace they span would be of dimension 2 hence would span $\mathbb{R}^2$. If two vectors were not independent, the subspace they span would be of dimension at most 1 and hence not all of $\mathbb{R}^2$.

4. Find the general solution to the coupled system of differential equations given by $\frac{d^2x}{dt^2} = \left(\begin{smallmatrix} -1 & 5 \\ 5 & -1 \end{smallmatrix}\right)x$. Hint: try $y_1=x_1+x_2, y_2=x_1-x_2$.

   You may leave the constants as constants, you need not find them in terms of $x(0)$.

   The suggested substitution leads to $\frac{d^2y}{dt^2} = \left(\begin{smallmatrix} 4 & 0 \\ 0 & -6 \end{smallmatrix}\right)y$, which has solutions $y_1(t) = c_1e^{2t}+c_2e^{-2t}, y_2(t)=c_3\cos(\sqrt{6}t)+c_4\sin(\sqrt{6}t)$.
   We have $x_1=(y_1+y_2)/2, x_2=(y_1-y_2)/2$, so (setting $d_i=c_i/2$) we get
   $x_1(t)=d_1e^{2t}+d_2e^{-2t}+d_3\cos(\sqrt{6}t)+d_4\sin(\sqrt{6}t)$,
   $x_2(t)=d_1e^{2t}+d_2e^{-2t}-d_3\cos(\sqrt{6}t)-d_4\sin(\sqrt{6}t)$.